Resilience of Single-Layer and Multiplex Networks Following Sudden Changes to Tie Costs

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I. INTRODUCTION

The formation of social ties is dictated by the incentives and opportunities to do so on the part of the individuals involved. Those individual incentives can shape the emergent structure of the networks that subsequently form. For example, when social ties are preferentially made with already well-connected individuals, the resulting networks exhibit a scale-free structure that is quite different from networks formed at random [1]. In the formation of social ties, many types of incentives may operate simultaneously, based on the psychology and economics of social connection, as well as the sociological benefits of participating in a rich network. These incentives include the raw costs or benefits of maintaining social relationships [2–5], the costs or benefits of closing triangles [6, 7], and the costs or benefits of having what have been called "spillover ties"—ties with the same individuals across multiple contexts, which, among other things, can save on transaction costs and provide new social affordances [8–12]. The kinds of network structures that result from such incentives acting in concert are important to understand, but have not been extensively studied, particularly for cases involving multiplex networks.

Moreover, incentives for social ties may not be constant over time. If the process of network formation is not strongly path dependent, then this inconstancy may not matter much; the structure of the network will reflect whatever incentives are currently driving individuals behavior. On the other hand, consider scenarios in which incentives at one time allow for the formation of structural elements of a social network that would not easily arise under different incentives. Once present, however, those structural features may be stable under new incentives, even though they could not have arisen *de novo* with those incentives. As a slightly more concrete example, imagine that low tie costs can facilitate the formation of closed triangles, which carry their own benefits. If tie costs subsequently increase due to external factors, the benefits received from the triangles already in place may offset the increased costs of maintaining those social ties. On the other hand, if tie costs had always been high, there may not ever have been sufficiently many edges to facilitate clustering. We refer to this type of phenomenon as *structural entrenchment*: the persistence of structural features formed under different conditions or incentives than those currently prevailing, which would not have formed had the current conditions always existed.

Our goal is to examine scenarios of exactly this type, and to establish a related body of formal theory. To do so, we study a dynamic model of social network formation on single-layer and multiplex networks with structural incentives that vary over time. Our goals are first, to examine the types of network structures that emerge from several incentives acting simultaneously, and second, to explore conditions under which network structures are or aren't resilient to changes in those incentives. In particular, we will examine a two-layer multiplex network on which incentives exist for social ties, closed triangles, and spillover ties. We will consider changes to incentives in the form of system-wide shocks, such that all individuals in the network experience identical changes to incentives, focusing on changes to the cost of social ties. Our model is not meant to reproduce any particular social system, but rather to intuit implications for a broad class of systems. Abstract models, even unrealistic ones, have proven quite valuable in forming intuitions of this sort [13–15].

A. Social Ties and Triangles

Social connections are incentivized in many ways. Social connections provide psychological and health benefits [3–5], opportunities for cooperation [12, 16, 17], learning [18–20], and economic activity [2, 21, 22]. As a rhetorical illustration, consider the characters in JK Rowling's *Harry Potter* series. Harry benefits from being friends with both Ron and Hermione. Ron gives him companionship, tells him about the wizarding world, and gives him a place to stay over the holidays. Hermione helps him with his homework and to develop a stronger sense of empathy. Thus, it is not a far stretch to imagine that

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social ties carry benefits. Obviously there are limits to how these benefits accrue, however. One cannot have 10,000 close friends (no matter what some avid social media fans claim), because of the cognitive, temporal, and pragmatic constraints to maintaining all of those relationships. Furthermore, the benefit to social relationships may have diminishing marginal returns. If you have no friends, making one is of tremendous importance. If you have 30 friends, however, adding a 31^{st} may carry few benefits unless your new friend brings something quite unique to the relationship. In our model, we will consider benefits to social ties with diminishing marginal returns. Although many factors influence the value of forming a social tie with one individual rather than another, for simplicity we will assume that, all else equal, the value of a social tie is insensitive to the identity of the individuals involved.

B. Multiplexity and Spillover Ties

The overwhelming majority of research on social networks has been on single-layer networks, defined by a set of nodes and a set of ties between them. Each set of ties is known as a layer. For many important problems, however, it is valuable to consider the multiplex nature of social networks. That is, for a given set of nodes (representing individuals), there may exist multiple contexts for each of which a different set of ties describes the structure of social relationships. Recently, a body of work has arisen to study formal properties of multiplex networks, which both extends traditional network theory to multiplex networks and also explores unique properties of networks with more than one layer and interdependencies between or among layers [23–31].

As an example of a multiplex, consider a set of individuals for whom we can construct a neighborhood network indicating residential contiguity among people. Two people are connected if they live on the same block. Consider also a friendship network in which people are connected if they are friends. Finally, consider an organizational network in which two people are connected if they participate together in formal social settings such as work or volunteer organizations. Individual behaviors on any of these networks are not necessarily independent of the other networks. You might become friends with your neighbors or the people you work with, and in doing so create opportunities that don't exist for friends who aren't neighbors or neighbors who aren't friends. Influence between layers of a multiplex network is sometimes known as *spillover.* In our model, we consider a spillover effect in a two-layer network: nodes get a boney from forming a tie with a node in one layer if they already have a tie with the same node in the other layer.

C. Changing Incentives

The costs for forming or maintaining ties may change dramatically over time. The relative cost to forming new social ties may be small for childless urban twenty-somethings, but rather high when some of them grow older, acquire demanding jobs, romantic partners, and children. Social relationships formed when younger and more carefree may become structurally entrenched by acquiring additional benefits, such as those enjoyed by a tight-knit group of friends who trade gossip on one another, that can outweigh the increased costs of maintaining relationships later in life when demands on one's time have increased.

As another example, trade agreements between corporations may form under supportive economic conditions, such as those enjoyed among EU nations, which may then become complicated when those conditions change, such as would occur if and when the United Kingdom exits the EU and restricts, among other things, the ability for UK citizens to take jobs in Europe. If, however, relationships are structurally entrenched, perhaps because they share suppliers or distributed, the benefits of remaining connected may outweigh the new costs.

We model changes to the cost of social ties, leaving constant the benefits of ties, triangles, and spillover ties. We refer to these changes as *shocks*, because they are sudden, system-wide changes to the system. We are interested both in shocks that *increase* costs—which may reduce the capacity of the network to maintain structure in the form of social ties—are well as in shocks that *decrease* costs—which may increase the capacity of the network to maintain such structure. We explore conditions under which the network exhibits resilience and maintains structure after a shock.

II. MODEL

Nodes represent individuals (or agents), and ties represent an ongoing social relationship between those individuals. For simplicity, all edges are assumed to be undirected and unweighted. Our model is adapted from a previous model by Burger and Buskens [32], who explored network formation on a single-layer network in response to incentives for ties and closed triangles. In their model, nodes in an empty network could bilaterally add ties when they increased the utility of both parties, and drop ties unilaterally if doing so would increase a node's utility. We extend this to a two-layer multiplex in which there can be additional incentives to spillover edges. We then examine network formation and explore the effects of exogenous *shocks*, which occur after the network has reached an equilibrium. A shock is operationalized here as system-wide change in the cost of social ties. Burger and Buskens [32] restricted their analysis to small six-node networks. Our analysis differs in that we consider networks of arbitrary size, N. Our dynamics also differ from theirs in that agents in our model are able to consider in their decisions the total utility resulting from rewiring—that is, simultaneously dropping one tie and adding a different tie— whereas their model required all individual add or drop actions to be utility-increasing. See below for details.

A. Utility

An agent's utility results from three aspects of the social structure of an individuals' local network. First, *ties* have intrinsic benefits and costs. Each agent receives a direct benefit for each tie it holds with another agent. However, maintaining ties is also costly due to constraints on time, attention, and transaction costs [32–34]. We assume that benefits accrue linearly with the number of ties, while the costs accrue at a faster rate. Our functional form therefore represents diminishing marginal returns to adding additional social ties. Other functional forms that accomplish similar diminishing marginal returns are of course possible.

Second, closing *triangles* may yield an additional benefit. We focus on scenarios in which local network closure is an important form of social capital, such as through reducing the costs of information search and facilitating the coordination on social norms [6, 7]. In other scenarios, closed triangles may be undesirable, as utility is gained through bridging structural holes [35]. Such scenarios are also of interest, but for simplicity we do not consider them in our analysis here.

Third, we consider the benefit of *spillover* ties across layers of the multiplex. Humans interact in multiple contexts that influence one another. Here, we consider scenarios in which having a tie with an individual in multiple layers (or contexts) carries an additional benefit. For example, being friends with your neighbor may carry benefits beyond the sum of benefits from having a friend and having a neighbor. We refer to the benefits and costs of ties, triangles, and spillover in aggregate as the *structural incentives* of the network. The basic assumption is that nodes act to maximize their marginal utility, that is, they choose ties that maximize the net benefits from their structural incentives.

Our analysis is restricted to a two-layer multiplex (Fig 1). We operationalize utility by extending the functional form introduced in Ref. [32] to a two-layer multiplex and including spillover benefits. The utility to agent i, with $t_{i\ell}$ ties and $z_{i\ell}$ closed triangles in each layer ℓ and v_i spillover ties is given by the following function:

$$u_i = \sum_{\ell \in \{1,2\}} \left(b t_{i\ell} - c t_{i\ell}^2 + d z_{i\ell} \right) + e v_i, \qquad (1)$$

where b and c are the benefits and costs of maintaining a tie in either layer, d is the benefit to a closed triangle in either layer, and e is the benefit of spillover ties.

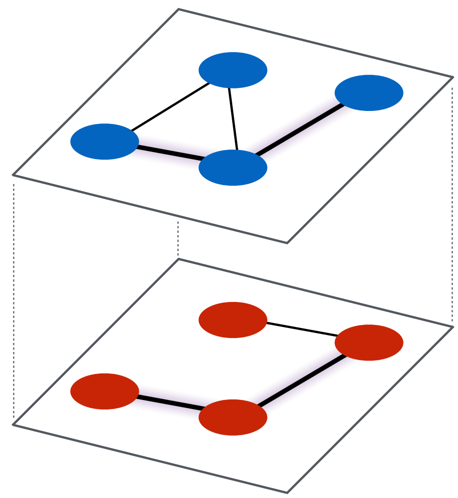

FIG. 1. A schematic of the model system, here shown as a four-node multiplex with two layers. The three leftmost nodes are part of a closed triangle in Layer 1 (blue) but not in Layer 2 (red). The three bottommost nodes have spillover ties (ties with the same nodes in both layers), depicted in bold.

B. Network formation dynamics

Agents add new ties and drop existing ties in order to increase their utility. Time is discrete and occurs in rounds. Each round, each agent has the opportunity to proactively add one new tie and delete one existing tie, though neither action is obligatory. We say "proactively," because agents may also gain or lose ties through the actions of others. At the beginning of each round, each agent, in random order, samples p other agents in the network.

On its move, an agent i considers all possible ties in each layer of the multiplex not currently held and identifies the tie with node j in layer ℓ whose addition would the largest increase in utility, $\Delta u^+_{ij\ell}$. If multiple ties have equally high value, one is selected at random. If $\Delta u^+_{ij\ell} > 0$, agent i proposes the tie. If $\Delta u^+_{ji\ell} > 0$, that is, if the addition of the tie would also increase j's utility, then the tie is formed, otherwise it is not. Agents can only propose one new tie each round, regardless of whether their proposal is accepted[1].

If the straightforward addition of any new tie will not increase the agent's utility, the agent then examines whether it could increase its utility by rewiring, considering only those p nodes sampled. In other words, could the agent increase its utility by dropping a currently held tie with node h and replacing it with a tie with node j?

[1] Our model assumes that nodes are not aware of the local networks and corresponding utilities of other nodes. If they were, they could selectively offer ties only to those nodes likely to accept them. This informational constraint is likely to apply for some systems and not others.

Here the agent considers all such pairings, and identifies the pair (h, j) such that dropping its existing edge with h and adding a new tie with j has the largest marginal utility. If that marginal utility is larger than zero, the agent proposes a tie with node j. If that tie is acceptable to j (i.e., it increases j's utility), the tie is made, and the agent then drops its edge with node h. Otherwise, no action is taken. The newly added tie need not be in the same layer as the dropped tie, corresponding to agents' ability to differentially allocate resources across contexts.

If no current tie has been dropped, the agent then considers all its current ties, excluding any just added, and identifies the tie for which dropping would lead to the largest marginal utility gain. If that gain is larger than zero, the agent drops the tie.

This process of network formation continues until a stable network equilibrium has been reached. We operationally define an equilibrium after five complete rounds in which no ties are added or dropped.

C. Noise

When choosing a new tie to propose, with probability ν an agent selects an (unconnected) node and layer at random, and such a proposal is accepted without regard for utility with the same probability. Similarly, an agent drops an existing tie at random with probability ν. Unless otherwise stated, simulations used $\nu = 0$.

D. Shocks

Once the network reaches a state of equilibrium, a shock occurs. A shock is an exogenous event that simultaneously changes tie costs for all agents. After a shock, new structural changes (i.e., adding new ties or dropping existing ties) may result in a utility increase for some agents.

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